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USE OF EIGENVALUE ANALYSIS FOR DIFFERENT LEVELS OF STABILITY DESIGN

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***Abstract**. Stability analysis and design has always played a key role in the process of verification of steel structures. The possible analysis methods and design procedures have long history with a plentiful literature providing various proposals for the engineers. This paper concentrates on the use of different types of eigenvalue analysis as a simple and powerful tool for stability design. Nowadays almost all the engineering software products have some kind of eigenvalue analysis options so these tools are easily available for the practicing engineers providing them a deeper look on the structural behavior. Various types of application possibilities are reviewed and new methods are proposed supporting the most up-to-date standard procedures of different levels from the isolated member design to the partial or global structural stability design. The suitable theoretical (both mathematical and mechanical) background is developed and the numerical procedure is implemented. The technique is applicable for a wide range of structural types and stability problems making the automatic effective length calculation possible in general without the use of any iterative process or tabulated values for certain cases. An application example is presented showing the comprehensiveness of the methods, and special efficiency indicators are presented in order to supply information about the adequacy of the applied design method.*

1 INTRODUCTION

There are two available methods for stability design of steel structures provided by the EN 1993-1-1 [1]:

- isolated member approach: Sections 6.3.1, 6.3.2, 6.3.3
- general method: Section 6.3.4

The first one is valid for uniform members only and based on structural *member isolation* and buckling *mode separation*. The main difficulties of this method arise from these two simplifications, the member isolation is usually handled by applying suitable effective length factors – considering the appropriate support and restraint conditions – while the mode separation is solved by special interaction factors creating the connection between the pure loading and buckling cases. There are a great number of papers on both topics including some theoretical investigations about the mechanical basics [2, 3] and several proposals for the practical application [4, 5]. The EN 1993-1-1 regularizes only the calculation of interaction factors the problems coming up from the member isolation are not dealt with in the standard. In the general method these two simplifications are eliminated by examining a complete structural part and calculating only one slenderness belonging to the real, compound loading and buckling situation. Although in the recent version of EN 1993-1-1 there are several restrictions on the application field of this method however on the other hand there are heavy research and development efforts on extending its applicability [6, 7] and this method is expected to cover much larger area of practical problems then the isolated member based conventional procedures. It is also important to note that in case of the general method the calculation of the generalized structural slenderness requires more complicated analysis which can usually provided only by application of some software package. It seems evident is that in both methods the key question is the determination of the slenderness values and especially the calculation of the appropriate elastic critical values (critical forces or critical load levels). This aspect is very poorly treated in the structural standards and accordingly the practicing engineer is fully responsible for it. Moreover the calculated elastic critical values are usually very important and have a significant influence on the final result of stability design.

In this paper the possibilities of eigenvalue analysis for the calculation of elastic critical values are examined from the point of view of standard stability design according to the recent version of EN 1993-1-1. Different application methods are introduced adapted to the different design approaches, and special indicator factors are developed highlighting the relevance of the used type of eigenvalue analysis. An application example is presented to show the practical working of the different methods.

2 METHODS OF EIGENVALUE ANALYSIS

For a usual steel structure composed of beam-column elements the general loss of elastic stability can be quite accurately described by bifurcation analysis. In a standard finite element environment this problem can be formulated as a linear eigenvalue analysis with the following basic form:

$$\left(\boldsymbol{K}_E - \alpha \boldsymbol{K}_G\right)\boldsymbol{U} = 0 \tag{1}$$

where $\boldsymbol{K}_E$ is the elastic stiffness matrix, $\boldsymbol{K}_G$ is the second order geometric stiffness matrix, α is the eigenvalue and $\boldsymbol{U}$ is the corresponding eigenvector. In the mechanical interpretation the eigenvalue denotes the elastic critical load level and the eigenvector shows the eigenshape (eigenmode) or buckling shape (buckling mode). It is important to note that the mechanical meaning and accuracy of the calculated eigenmodes and eigenvalues highly depend on the definition of stiffness matrices. In this paper special decomposition techniques are applied for the compiled geometric stiffness matrix – while the elastic stiffness matrix is always formed on the complete structural model – to suitably calculate the elastic critical load levels necessary for the certain design approaches. Upon these techniques the following eigenvalue analysis are proposed:

- Complete Eigenvalue Analysis (CEA) – the geometric stiffness matrix is compiled on the whole structural model
- Partial Eigenvalue Analysis (PEA) – the geometric stiffness matrix is compiled only on a separated part of the structural model
- Selected Eigenmode Analysis (SEA) – the geometric stiffness matrix is compiled only on selected displacement degree-of-freedoms

The CEA is the mostly known and commonly used analysis technique, the resulted critical load levels and corresponding buckling modes apply to the whole global structural model – even it is apparently restricted to a part of it – and consider the compound loading case. In the further text we write the CEA in the following form:

$$\left(\boldsymbol{K}_E - \alpha_{cr}^{C} \boldsymbol{K}_G^{C}\right)\boldsymbol{U}_{cr}^{C} = 0 \tag{2}$$

The CEA method is applicable for the critical load level calculation for the general stability design method.

If a part of the complete model is examined and intended to design for stability the PEA can be used. In this method the structural model is divided into two parts: (P1) a relevant part and (P2) a remaining part. Accordingly the complete geometric stiffness matrix can be decomposed:

$$\boldsymbol{K}_G^{C} = \boldsymbol{K}_G^{P1} + \boldsymbol{K}_G^{P2} \tag{3}$$

One can obtain an eigenvalue solution for the relevant part by solving the following equation:

$$\left(\boldsymbol{K}_E - \alpha_{cr}^{P1}\boldsymbol{K}_G^{P1}\right)\boldsymbol{U}_{cr}^{P1} = 0 \tag{4}$$

In a mechanical interpretation this calculation yields special buckling shapes which are induced by the internal forces acting only in the members which are part of the relevant substructure while the initial stiffness of the whole structure is considered (as a restraint condition for the examined substructure). In that sense this solution is similar to the ones applying fictitious springs at the joining parts of the relevant substructure modeling the restraints from the remainder of the whole structure. The PEA method also provides global type solution for the real, combined buckling situation so it is also applicable for the general stability design.

The third proposed method (SEA) is directly developed for the isolated member approach since it is able to calculate the separated buckling modes. In this case the geometric stiffness matrix is compiled in such a way that the rows and columns associated with the displacement degree-of-freedom necessary for the relevant buckling mode are considered and the rest of the matrix is neglected. For example if the subject of the analysis is the lateral buckling about the minor axis (axis ‘z’ according to EN 1993-1-1) then those terms are kept only in the geometric stiffness which contain the second order compression effect for the selected displacements (u_y and ϕ_z). Further if this buckling mode is required only on an isolated member then this reduced compilation is done only on this element. Consequently the second order effects of the system are concentrated so as to be able to experience only the relevant buckling mode. Considering however the complete first order elastic stiffness matrix of the whole structure the appropriate restraints (and accordingly the necessary effective lengths) of the isolated member can be calculated quite accurately. The SEA method for the selected *i*-th buckling mode (for instance $N_{cr,z}$) is written as follows:

$$\left(\boldsymbol{K}_E - \alpha_{cr}^{\mathrm{Ncr,z}}\boldsymbol{K}_G^{\mathrm{Ncr,z}}\right)\boldsymbol{U}_{cr}^{\mathrm{Ncr,z}} = 0 \tag{5}$$

3 DEVELOPMENT OF EIGENMODE RELEVANCE INDICATOR (ERI)

After introducing the proposed possibilities for the determination of elastic critical load levels by applying certain eigenvalue analysis methods the next important issue is the selection of the most appropriate method for the current structural problem examined. It is very important to detect the most relevant buckling modes of the structure and the associated most proper design method. The introduced eigenvalue analysis methods can yield various results and solution possibilities but it is the decision of the engineer which method and buckling mode is the most relevant for the problem. In order to help this decision special indicator factors are developed showing the relevance of the calculated buckling modes (eigenmode relevance indicator – ERI). The ERIs are formulated on energy base, this approach was introduced to the interpretation of stability calculations in [8], however, for different purposes; in this paper the ERIs are developed so as to supply appropriate information about the described issues. The basic formula of all the possible ERIs is the internal (and the corresponding external) energy induced by the *i*-th eigenmode; this can be written using Eq. (2):

$$E_i = \frac{1}{2}\boldsymbol{U}_{cr,i}^T\boldsymbol{K}_E\boldsymbol{U}_{cr,i} = \frac{1}{2}\alpha_{cr,i}\boldsymbol{U}_{cr,i}^T\boldsymbol{K}_G\boldsymbol{U}_{cr,i} \tag{6}$$

The next sections present the ERIs in case of the different eigenvalue analysis methods based on Eq. (6).

3.1 CEA

When analyzing the complete structure for stability the following questions may arise regarding the obtained eigenmodes:

(1) for a certain eigenmode what are the relevant members (relevant model portion)?
(2) for certain members (certain model portion) which is the most relevant eigenmode?

For the problem (1) let the energy be calculated for members or model portions (k denotes the relevant member or model portion):

$$E_i^k = \frac{1}{2}\boldsymbol{U}_{cr,i}^C \boldsymbol{K}_E^k \boldsymbol{U}_{cr,i}^C \tag{7}$$

In this case the elastic stiffness of the relevant model part is considered only. Obviously the sum of all the energy values of model parts gives the total energy of the complete model, i.e. (having m number of model parts):

$$E_i = \sum_{k=1}^{m} E_i^k \tag{8}$$

Accordingly an ERI can be constructed so as to show the relevance of the separated model portions considering the i-th eigenmode as a percentage:

$$ERI1_i^k = 100\frac{E_i^k}{E_i} = 100\frac{\boldsymbol{U}_{cr,i}^C \boldsymbol{K}_E^k \boldsymbol{U}_{cr,i}^C}{\boldsymbol{U}_{cr,i}^C \boldsymbol{K}_E \boldsymbol{U}_{cr,i}^C} \tag{9}$$

Problem (2) is more complicated but also more significant, since in the case of a complex structural model it is usual, that different eigenmodes describe the buckling behavior of distinct parts of the model. For that reason a scaling procedure is necessary in order to select the appropriate eigenmode for the stability design. To develop a proper scaling factor let us examine the basis of the stability design approach of the EN 1993-1-1 which is the buckling curve based reduction factor. The mechanical model for the buckling reduction factors is the Ayrton-Perry formula. In this model the failure is associated with the load level at which the second order maximum elastic stress of the geometrically imperfect member reaches the yield stress. Consequently the reduction factor depends mainly on the amplified imperfection which has usually a shape equal to an appropriate eigenmode or a combination of them. The scaling factor should therefore consider this effect to show the importance of the eigenmodes in accordance with the mechanics of the buckling reduction factors. Firstly the basic amplitudes for the eigenmodes are determined by normalizing using the elastic stiffness matrix:

$$E_i = \frac{1}{2}\boldsymbol{U}_{cr,i}^N \boldsymbol{K}_E \boldsymbol{U}_{cr,i}^N = 1 \Rightarrow \frac{\boldsymbol{U}_{cr,j}^N \boldsymbol{K}_G^C \boldsymbol{U}_{cr,j}^N}{\boldsymbol{U}_{cr,i}^N \boldsymbol{K}_G^C \boldsymbol{U}_{cr,i}^N} = \frac{\alpha_{cr,i}^C}{\alpha_{cr,j}^C} \tag{10}$$

As a result the greater the critical factor the less the amplitude of the eigenmode is, this is realistic when considering the eigenmodes as geometric imperfection. It is known [6] that the geometrical imperfections having the shape of an eigenmode cause the following additional amplified second order displacements:

$$\boldsymbol{U}_{add} = \frac{1}{\alpha_{cr} - 1}\boldsymbol{U}_{imp} \tag{11}$$

If the normalized eigenmodes of Eq. (10) are considered as imperfections in Eq. (11) then a further scaling factor can be created by calculating the energy of this amplified imperfection in the k-th model portion:

$$E_i^k = \frac{1}{2}\left(\frac{1}{\alpha_{cr,i} - 1}\boldsymbol{U}_{cr,i}^N\right)^T \boldsymbol{K}_E^k \left(\frac{1}{\alpha_{cr,i} - 1}\boldsymbol{U}_{cr,i}^N\right) \tag{12}$$

Finally if it is considered that enough number (m) of eigenmodes is calculated (i.e. the last eigenvalue – what is the highest elastic critical load level – is sufficiently high) then the following ERI can be constructed showing the relative significance of the i-th eigenmode for the k-th model portion:

$$ERI2_i^k = 100\frac{E_i^k}{\sum_{i=1}^{m} E_i^k} = 100\frac{\left(\frac{1}{\alpha_{cr,i}-1}\right)^2 \boldsymbol{U}_{cr,i}^N \boldsymbol{K}_E^k \boldsymbol{U}_{cr,i}^N}{\sum_{i=1}^{m}\left(\frac{1}{\alpha_{cr,i}-1}\right)^2 \boldsymbol{U}_{cr,i}^N \boldsymbol{K}_E^k \boldsymbol{U}_{cr,i}^N} \tag{13}$$

3.2 PEA and SEA

In this case the determination of the relevant model portion is not examined since the preliminary selection of model part (or isolated member) aims at concentrating the buckling mode to this part. What is important however that how accurate the calculated partial buckling mode is compared to the possible complete modes. Let us write the eigenvalue equation for the complete structure (Eq. (2)) using the partial eigenmode and eigenvalue yielded by the solution of Eq. (4):

$$\left(\boldsymbol{K}_E - \alpha_{cr,i}^{P1}\boldsymbol{K}_G^C\right)\boldsymbol{U}_{cr,i}^{P1} = \left(\boldsymbol{K}_E - \alpha_{cr,i}^{P1}\boldsymbol{K}_G^{P1}\right)\boldsymbol{U}_{cr,i}^{P1} - \alpha_{cr,i}^{P1}\boldsymbol{K}_G^{P2}\boldsymbol{U}_{cr,i}^{P1} = -\alpha_{cr,i}^{P1}\boldsymbol{K}_G^{P2}\boldsymbol{U}_{cr,i}^{P1} \tag{14}$$

Eq. (14) expresses an error force vector generated by the partial buckling mode on the remaining model portion (P2). Obviously if this term is significant then the partial buckling mode is possibly irrelevant which shows that the current model portion has no distinct buckling problem. On the other hand if this force approaches zero then the buckling mode is dominant and accurate for the selected model portion. To develop a straightforward indicator we use again an energy format for Eq. (14):

$$ERI3_i^{P1} = 100\left(1 - \frac{\boldsymbol{U}_{cr,i}^{P1}\boldsymbol{K}_G^{P2}\boldsymbol{U}_{cr,i}^{P1}}{\boldsymbol{U}_{cr,i}^{P1}\boldsymbol{K}_G^{C}\boldsymbol{U}_{cr,i}^{P1}}\right) = 100\frac{\boldsymbol{U}_{cr,i}^{P1}\boldsymbol{K}_G^{P1}\boldsymbol{U}_{cr,i}^{P1}}{\boldsymbol{U}_{cr,i}^{P1}\boldsymbol{K}_G^{C}\boldsymbol{U}_{cr,i}^{P1}} \tag{15}$$

4 APPLICATION EXAMPLE

In the following example the CEA method is used and the ERI1 and ERI2 factors are examined.

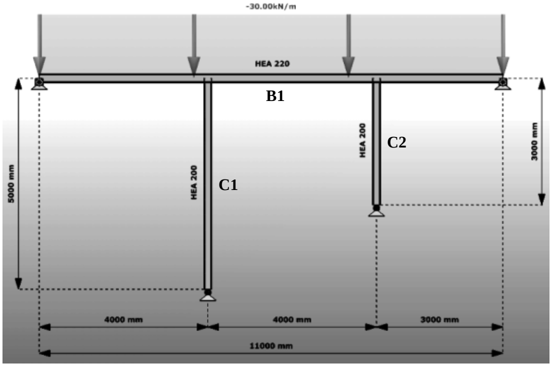

Figure 1: Model of the example (made in ConSteel [10]).

In Fig. 1 the structural model is illustrated, which consists of two columns (C1 and C2) of HEA200 and a beam (B1) of HEA220 loaded by a 30 kN/m line load. According to the support condition four cases are evaluated (the beam is always simply supported) summarized by Table 1.

Table 1: Evaluated support conditions.

	C1	C2
Case1	Pinned	Pinned
Case2	Fixed	Pinned
Case3	Pinned	Fixed
Case4	Fixed	Fixed

All the eigenvalue analysis are calculated by ConSteel software [9], [10] using a 7 degree-of-freedom finite element model. Figs. 2-5 show the calculated eigenmodes and elastic critical factors for Case1-Case4 – all buckling modes form some kind of out-of plane buckling –, and Tables 2-5 contain the ERI1 and ERI2 factors evaluated for the beam and the two columns (for ERI1 the columns and for ERI2 the rows give the 100% value).

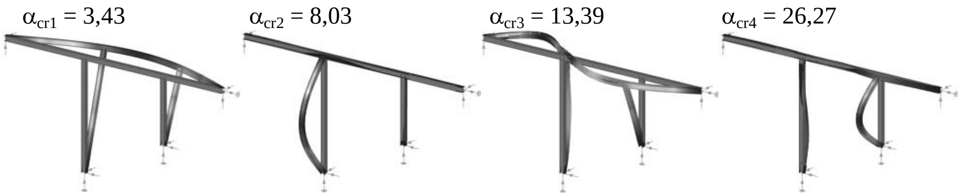

Figure 2: Case1 – first four eigenmode and eigenvalue.

Table 2: Case1 – ERI1 and ERI2 factors

	ERI1				ERI2			
	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}
B1	99,6	2,4	95,4	7,6	96,1	0,3	3,5	0,1
C1	0,1	97,5	4,4	12,9	0,5	97,1	1,4	1,0
C2	0,3	0,1	0,2	79,5	26,9	0,4	0,6	72,1

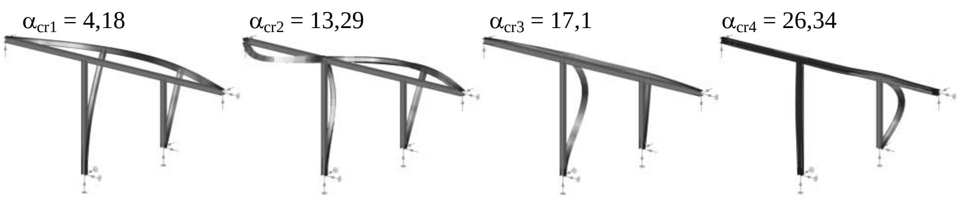

Figure 3: Case2 – first four eigenmode and eigenvalue.

Table 3: Case2 – ERI1 and ERI2 factors

	ERI1				ERI2			
	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}
B1	73,1	91,7	10,1	5,2	91,7	7,7	0,5	0,1
C1	26,7	8,2	89,6	0,2	86,8	1,8	11,4	0,0
C2	0,2	0,1	0,3	94,6	13,9	0,4	0,5	85,2

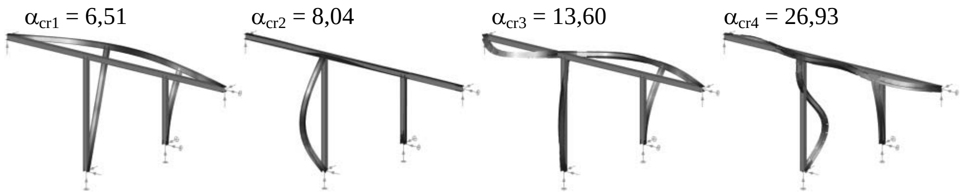

Figure 4: Case3 – first four eigenmode and eigenvalue.

Table 4: Case3 – ERI1 and ERI2 factors

	ERI1				ERI2			
	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}
B1	54,1	2,4	91,5	9,4	73,6	2,0	23,8	0,6
C1	0,0	97,5	4,5	90,4	0,1	92,3	1,3	6,3
C2	45,9	0,1	4,0	0,2	98,3	0,0	1,6	0,1

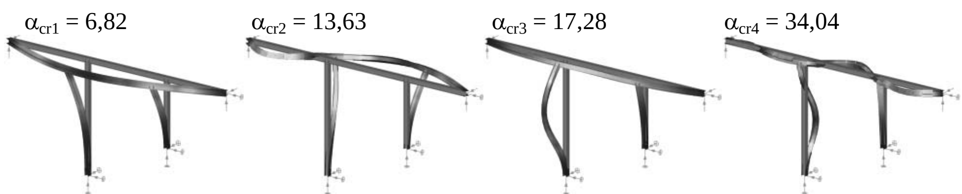

Figure 5: Case4 – first four eigenmode and eigenvalue.

Table 5: Case4 – ERI1 and ERI2 factors

	ERI1				ERI2			
	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}	α_{cr1}	α_{cr2}	α_{cr3}	α_{cr4}
B1	43,3	89,0	8,1	76,0	66,0	28,8	1,6	3,6
C1	20,0	5,4	89,9	19,8	60,1	3,4	34,6	1,9
C2	36,7	5,6	1,9	4,2	95,9	3,1	0,7	0,3

It can be seen that the first and dominant buckling mode is always the lateral-torsional buckling of the B1 beam coupled with some form of flexural buckling of the columns. The corresponding critical load factors increases from Case1 to Case4 denoting the significance of the applied additional restraints for the supports of the columns however from the ERI factors it becomes clear that the contribution of the columns to the certain buckling modes is significantly different. From the ERI1 values it can be concluded that in the first buckling mode the B1 beam is always dominant and for the other modes the dominant member is also highlighted in the tables. Looking at however the ERI2 values the followings can be determined for a certain member:

- which is the most relevant buckling mode;
- which is the appropriate critical load factor for the calculation of member slenderness in the stability design.

In Case1 the ERI2 values show the strong dominance of the member B1 in the first mode, C1 in the second mode and C2 in the fourth mode. This is a consequence of the pure shape of the buckling modes, since the columns have pinned supports. In Case2 the fix support of the column C1 has a considerable restraining effect on the beam increasing the first critical load factor accordingly in this case the first mode is more dominant for the column C1 than the third one which show an isolated buckling mode for this member. The column C2 is not really effected by the additional restraint, so the dominant mode remains the fourth one with almost the same critical load factor value. In Case3 the situation is quite the same, the column C1 has the dominant isolated buckling mode with same critical load factor as in Case1, and for column C2 the first mode is the most relevant. In Case4 from the ERI1 factors it is clear that the complete model contributes to the first buckling mode and the ERI2 values are explicitly show that this mode is the most relevant for all the members. Naturally the meaning of the ERI2 values can be refined by increasing the number of calculated eigenmodes which is usually necessary in case of larger structural models. This simple and straightforward example is intended to show the mechanical meaning of the different indicator factors.

5 CONCLUSIONS

One of the most important issues in stability design which is out of the field of standard regulations is the calculation of elastic critical forces or load levels for the determination of slenderness values. In this paper several methods are presented for this problem using the eigenvalue analysis based approaches. For different structural arrangements different types of buckling modes can be dominant and moreover the modern structural standards provide several different possibilities for the stability design. To yield appropriate slenderness for the different problems and design methods three approaches are proposed: the complete eigenvalue analysis, the partial eigenvalue analysis and the selected eigenmode analysis. Further supporting the selection of the relevant approach and eigenmode special indicator factors are developed. In an application example the practical working with the proposed eigenvalue analysis approaches and indicator factors are presented.

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